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On the Behavior of the Strong Unicity Constant for Changing Dimension

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1. INTRODUCTION

Let $C(I)$ denote the set of continuous, real valued functions on the interval $I = [-1, 1]$, and let $\mathscr{P}_{n+1} \subseteq C(I)$ be a Haar subspace of dimension $n + 1$. Denote the uniform norm on $C(I)$ by $\|\cdot\|$. For $f \in C(I)$ with best approximation $B_n(f)$ from $\mathscr{P}_{n+1}$ there is a positive constant r such that for any $p \in \mathscr{P}_{n+1}$,

$$\| p - B_n(f) \| \leqslant r(\| f - p \| - \| f - B_n(f) \|). \tag{1.1}$$

Inequality (1.1) is the well-known strong unicity theorem [3, p. 80]. The *strong unicity constant* $M_n(f)$ is defined to be the smallest constant r such that (1.1) is valid for all $p \in \mathscr{P}_{n+1}$.

The dependence of $M_n(f)$ on f, n, and I has been the subject of several recent papers [1, 4, 5, 6, 7, 9, 10]. The present paper is concerned with the dependence of $M_n(f)$ on n. Of the references mentioned above, [4, 6, 9, 10] examine the behavior of the sequence

$$\{M_n(f)\}_{n=0}^{\infty}. \tag{1.2}$$

The problem of characterizing those functions $f \in C(I)$ for which the sequence (1.2) is bounded is posed by Poreda [9]. Poreda constructs a function $f \in C(I)$ for which $\lim_n \sup M_n(f) = +\infty$. Henry and Roulier [6] demonstrate a class of functions $F \subseteq C(I)$ for which $\lim_n M_n(f) = +\infty$ for each $f \in F$. Henry and Roulier also conjecture that the sequence (1.2) is bounded only if f is a polynomial function. Schmidt [10] enlarges the class F for which $\lim_n M_n(f) = +\infty$, and proves that there exists a function $g \in C(I)$ for which

$$\liminf_n M_n(g) = 1, \qquad \limsup_n M_n(g) = +\infty. \tag{1.3}$$

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Although in some sense Schmidt actually constructs the function g satisfying (1.3), in reality the function is neither explicit nor easily analyzed.

In Section 3 of the present paper an explicit function satisfying (1.3) is given and analyzed.

Cline [4] examines the order of the strong unicity constant for $f \in C(X)$, X finite. If $\mathscr{P}_{N+1} = \Pi_N$, the set of polynomials of degree at most N, and if $p(x) = x^{N+1}$, then Cline [4] proves for an appropriate finite set $X \subseteq I$ that $M_N(p) = 2N + 1$. Thus Cline establishes the exact order of $M_n(p)$ for a particular n, namely $n = N$. We note for $n > N$ that $M_n(p) = 1$. Hence the precise order of $M_n(p)$ is known for every $n \geqslant N$.

In the next section the concept "precise order of $M_n(f)$" will be considered. In this regard, let $f \in C(I)$, and suppose there exist positive constants α and β, a natural number N, and a positive real valued function c with domain the natural numbers satisfying

$$\alpha c(n) \leqslant M_n(f) \leqslant \beta c(n) \tag{1.4}$$

for all $n \geqslant N$. Then the *precise order of* $M_n(f)$ *is* $0(c(n))$ for n sufficiently large.

To date the precise order of $M_n(f)$ has not been established for any non-polynomial function $f \in C(I)$.

The next section is devoted to showing that the precise order of the strong unicity constant $M_n(f)$ for the function $f(x) = 1/(x - a)$, $a \geqslant 2$, $x \in I$, and $\mathscr{P}_{n+1} = \Pi_n$, is $0(n)$.

2. Precise Orders

Let $f \in C(I)$, $f \notin \mathscr{P}_{n+1}$, and define $S(\mathscr{P}_{n+1}) = \{p \in \mathscr{P}_{n+1}: \| p \| = 1\}$. Then it is known [1, 2] that

$$M_n(f) = \{\inf_{p \in S(\mathscr{P}_{n+1})} \max_{x \in E_{n+1}(f)} \operatorname{sgn}[f(x) - B_n(f)(x)]\, p(x)\}^{-1}, \tag{2.1}$$

where

$$E_{n+1}(f) = \{x \in I: | f(x) - B_n(f)(x)| = \| f - B_n(f) \|\}. \tag{2.2}$$

Hereafter $\mathscr{P}_{n+1} = \Pi_n$. The first theorem is due to Cline [4], and will be utilized in the subsequent analysis.

THEOREM 1. *Let* $f \in C[-1, 1]$ *with* $f \notin \Pi_n$. *Let* $B_n(f) \in \Pi_n$ *be the best approximation to* f, *and for any Chebyshev alternation* $\{x_{kn}\}_{k=0}^{n+1}$ *for* $f - B_n(f)$, *define* $q_{in} \in \Pi_n$ *by* $q_{in}(x_{kn}) = \operatorname{sgn}[f(x_{kn}) - B_n(f)(x_{kn})]$, $k = 0, 1,..., n + 1$; $k \neq i$, *and* $i = 0, 1,..., n + 1$. *Then* $M_n(f) \leqslant \max_{0 \leqslant i \leqslant n+1}\{\| q_{in} \|\}$.

Henry and Roulier [6, p. 88] observe that if $E_{n+1}(f)$ contains exactly $n+2$ points, then the conclusion of Theorem 1 becomes

$$M_n(f) = \max_{0 \leqslant i \leqslant n+1} \{\| q_{in} \|\}. \tag{2.3}$$

The next theorem is an extension of the precise order results of Cline alluded to in section 1.

THEOREM 2. *Suppose that f is a polynomial of degree exactly $N+1$, and that $B_n(f) \in \Pi_N$ is the best approximation to f. Then the precise order of $M_n(f)$ is $0(c(n))$, where $c(N) = N$ and $c(n) = 1$ for $n > N$.*

Proof. We need only show that there exist positive constants α and β such that

$$\alpha N \leqslant M_N(f) \leqslant \beta N.$$

Let $f(x) = a_{N+1} g_{N+1}(x) + p_N(x)$, where $g_{N+1}(x) = x^{N+1}$ and $p_N \in \Pi_N$. Then $B_N(f)(x) = a_{N+1} B_N(g_{N+1})(x) + p_N(x)$ without loss of generality assume that $a_{N+1} > 0$, and let

$$e_N(f)(x) = f(x) - B_N(f)(x).$$

Then it is well known that

$$e_N(f)(x) = \frac{a_{N+1}}{2^N} C_{N+1}(x),$$

where C_{N+1} is the Chebyshev polynomial of degree $N+1$. Furthermore, the set of extreme points $E_{N+1}(f)$ is precisely the $N+2$ extreme points of C_{N+1}. Therefore the polynomials $\{q_{iN}\}_{i=0}^{N+1}$ defined in Theorem 1 satisfy

$$\frac{e_N(f)(x)}{\| e_N(f) \|} - q_{iN}(x) = 2^N \prod_{\substack{k=0 \\ k \neq i}}^{N+1} (x - x_{kN}), \tag{2.4}$$

where $E_{N+1}(f) = \{x_{kN}\}_{k=0}^{N+1}$ and where (2.4) follows from the classical remainder theorem of interpolation theory [3, p. 60]. Equation (2.4) may be rewritten as

$$q_{iN}(x) = C_{N+1}(x) - \frac{(x^2 - 1)\, C'_{N+1}(x)}{(N+1)(x - x_i)}, \tag{2.5}$$

$x \neq x_i$ and $i = 0, \ldots, N+1$. Hereafter if $h(x^*) = 0$ and if $\bar{h}(x) = h(x)/(x - x^*)$, $x \neq x^*$, then $\bar{h}(x^*)$ is defined to be equal to $h'(x^*)$. Equality (2.5) now implies that

$$q_{iN}(x) = C_{N+1}(x) + \left[\frac{(1 - x^2)\, C'_{N+1}(x) - (1 - x_i^2)\, C'_{N+1}(x_i)}{(N+1)(x - x_i)}\right],$$

and consequently an application of the mean value theorem yields

$$q_{iN}(x) = C_{N+1}(x) + \left[\frac{C''_{N+1}(\epsilon)(1 - \epsilon^2) - 2\epsilon C'_{N+1}(\epsilon)}{N + 1}\right],$$

for some ϵ between x and x_i. This last equality implies that

$$q_{iN}(x) = C_{N+1}(x) - \left[\frac{\epsilon C'_{N+1}(\epsilon) + (N + 1)^2 C_{N+1}(\epsilon)}{N + 1}\right],$$

which in turn implies that

$$|q_{iN}(x)| \leqslant \bar{\beta}N, \tag{2.6}$$

$i = 0, 1,..., N + 1$, and where $\bar{\beta}$ is independent of N. On the other hand, (2.5) implies that

$$q_{iN}(x_i) = C_{N+1}(x_i) + \frac{(1 - x_i^2)\, C''_{N+1}(x_i)}{N + 1},$$

$i = 1,..., N$. Therefore

$$\begin{aligned} q_{iN}(x_i) &= C_{N+1}(x_i) - (N + 1)\, C_{N+1}(x_i) \\ &= -NC_{N+1}(x_i), \end{aligned} \tag{2.7}$$

$i = 1, 2,..., N$. Finally, (2.6) and (2.7) combine to establish that

$$\bar{\alpha}N \leqslant \| q_{iN} \| \leqslant \bar{\beta}N, \qquad i = 1, 2,..., N,$$

where $\bar{\alpha}$ is also independent of N. Slight modifications in the above arguments produce similar bounds for q_{0N} and $q_{N+1,N}$. Therefore there exists positive constants α and β such that

$$\alpha N \leqslant \| q_{iN} \| \leqslant \beta N, \qquad i = 0, 1,..., N + 1.$$

An application of equality (2.3) completes the proof.

The next theorem is the main theorem of the present section.

THEOREM 3. *Let $f(x) = 1/(x - a)$, where $x \in I$ and $a \geqslant 2$, and let $B_n(f) \in \Pi_n$ be the best approximation to f, $n = 0, 1, 2,...$. Then for $n \geqslant 1$ the precise order of $M_n(f)$ is* $O(n)$.

Since $f^{(n)}(x) \neq 0$ for any $x \in I$, $n = 0, 1,...$, the extremal set (2.2) contains exactly $n + 2$ points. Thus equality (2.3) is valid, $n = 0, 1, 2,...$. Consequently to prove Theorem 3 it is sufficient to establish that there exists positive constants α and β such that

$$\alpha n \leqslant \max_{0 \leqslant i \leqslant n+1} \| q_{in} \| \leqslant \beta n, \tag{2.8}$$

for all $n \geqslant 1$.

The proof of Theorem 3 will be accomplished through a series of lemmas. For each lemma, it is assumed that the hypotheses of Theorem 3 are satisfied.

LEMMA 1. *Let the alternating set* $E_{n+1}(f)$ *be labeled* $-1 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$. *Define* $Q_{n+1} \in \Pi_{n+1}$ *to be the unique interpolating polynomial defined by*

$$Q_{n+1}(x_k) = (-1)^{n+k}, \qquad k = 0, 1,..., n+1. \tag{2.9}$$

Then for $n \geqslant 1$ *the* q_{in} *defined in Theorem* 1 *is given by*

$$q_{in}(x) = \left[Q_{n+1}(x) - \frac{a_{n+1} g(x)}{2^{n-1} n((a^2-1)^{1/2} + a)(x - x_i)}\right], \tag{2.10}$$

$i = 0, 1,..., n+1$, *where* a_{n+1} *is the leading coefficient of* Q_{n+1}, *and where*

$$g(x) = (x^2 - 1)(n(a^2-1)^{1/2} C_n(x) + (ax - 1) C_n'(x)). \tag{2.11}$$

Proof. As required in Theorem 1, we verify that q_{in} is the unique element of Π_n satisfying

$$q_{in}(x_k) = \operatorname{sgn}[f(x_k) - B_n(f)(x_k)], \tag{2.12}$$

$k = 0, 1,..., n+1$; $k \neq i$, $i = 0, 1,..., n+1$.

If $e_n(f)(x) = f(x) - B_n(f)(x)$, then it is known [8, 12] that

$$e_n(f)(x) = \frac{(a - (a^2-1)^{1/2})^n}{(a^2 - 1)} \cos(n\theta + \delta), \tag{2.13}$$

where $\cos \theta = x$ and

$$\cos \delta = \frac{ax - 1}{x - a}. \tag{2.14}$$

Therefore

$$\frac{e_n(f)(x_k)}{\| e_n(f) \|} = (-1)^{n+k}. \tag{2.15}$$

Comparing equality (2.15) with (2.12) establishes that q_{in} must satisfy

$$q_{in}(x_k) = (-1)^{n+k}, \tag{2.16}$$

$k = 0,\dots, n+1;\ k \neq 1,\ i = 0, 1,\dots, n+1$. Thus

$$q_{in}(x) = [Q_{n+1}(x) - a_{n+1}(x-x_0)\cdots(x-x_{i-1})(x-x_{i-1})\cdots(x-x_{n+1})] \tag{2.17}$$

It is also known [8] that the extremal set $E_{n+1}(f)$ consists of precisely the points -1, $+1$ and the n zeros of the polynomial

$$n(a^2-1)^{1/2}\, C_n(x) + (ax-1)\, C_n'(x). \tag{2.18}$$

This observation and (2.17) now imply (2.10).

The next three lemmas establish that $\| q_{in} \| \leqslant \beta n$.

LEMMA 2. *Let $Q_{n+1} \in \Pi_{n+1}$ be the unique interpolating polynomial satisfying* (2.9). *Then* $\| Q_{n+1} \| = O(n)$ *for* $n \geqslant 1$.

Proof. It is known [8] that

$$(ax-1)\, C_n(x) + \frac{1}{n}(a^2-1)^{1/2}(x^2-1)\, C_n'(x) = (x-a)\,\frac{e_n(f)(x)}{\| e_n(f)\|}. \tag{2.19}$$

Define $\bar{Q}_{n+1}$ by

$$\bar{Q}_{n+1}(x) = (ax-1)\, C_n(x) + \frac{1}{n}(a^2-1)^{1/2}(x^2-1)\, C_n'(x). \tag{2.20}$$

Then from (2.15) and (2.19) we have that

$$\bar{Q}_{n+1}(x_k) = (x_k - a)(-1)^{n+k}, \qquad k = 0, 1,\dots, n+1, \tag{2.21}$$

and (2.19) implies that $| \bar{Q}_{n+1}(x) | < a - x$ for $x \in I - E_{n+1}(f)$. Now Q_{n+1} can be written

$$Q_{n+1}(x) = \sum_{k=0}^{n+1} \frac{(-1)^{n+k}\,\omega(x)}{(x-x_k)\,\omega'(x_k)}, \tag{2.22}$$

where as usual

$$\omega(x) = (x-x_0)(x-x_1)\cdots(x-x_n)(x-x_{n+1}). \tag{2.23}$$

Since $x_1,\dots, x_n$ are the zeros of (2.18), (2.23) becomes

$$\omega(x) = (x^2-1)\left[\frac{n(a^2-1)^{1/2}\, C_n(x) + (ax-1)\, C_n'(x)}{n\, 2^{n-1}((a^2-1)^{1/2} + a)}\right]. \tag{2.24}$$

Let $A_n = n2^{n-1}((a^2-1)^{1/2}+a)$. Then (2.20) and (2.24) imply that

$$A_n\omega'(x) = n^2\bar{Q}_{n+1}(x) + 2x[n(a^2-1)^{1/2}\,C_n(x) + (ax-1)\,C_n'(x)] + (x-a)\,C_n'(x). \tag{2.25}$$

Since $\|\bar{Q}_{n+1}\| = a+1$ and $\|C_n'\| = n^2$, (2.25) yields

$$A_n\|\omega'\| \leqslant 2n[2(a+1)n + (a^2-1)^{1/2}]. \tag{2.26}$$

Evaluating (2.25) at $x = x_k$, employing (2.21), and utilizing the fact that the x_k are the zeros of (2.18), $k = 0, 1,\ldots, n+1$, yields

$$A_n\omega'(x_k) = (x_k - a)[n^2(-1)^{n+k} + C_n'(x_k)]. \tag{2.27}$$

From (2.19), (2.15), and (2.18) we have that

$$(ax_k - 1)\,C_n(x_k) + \frac{1}{n}(a^2-1)^{1/2}\,(x_k{}^2-1)\,C_n'(x_k) = (x_k - a)(-1)^{n+k},$$

and that

$$n(a^2-1)^{1/2}\,C_n(x_k) + (ax_k-1)\,C_n'(x_k) = 0,$$

$k = 1, 2,\ldots, n$. Eliminating $C_n(x_k)$ from these two equations results in

$$C_n'(x_k) = (-1)^{n+k}\,\frac{n(a^2-1)^{1/2}}{a - x_k}.$$

Substituting this expression into (2.27) produces

$$A_n\omega'(x_k) = (-1)^{n+k}[(x_k-a)\,n^2 - n(a^2-1)^{1/2}]. \tag{2.28}$$

Thus for $k = 1, 2,\ldots, n$,

$$A_n\,|\,\omega'(x_k)| \geqslant (a-1)\,n^2 + (a^2-1)^{1/2}. \tag{2.29}$$

On the other hand, direct substitution into (2.25) results in

$$A_n\omega'(1) = 2n[(a-1)n + (a^2-1)^{1/2}] \tag{2.30}$$

and

$$A_n\omega'(-1) = 2(-1)^{n+1}n[(a+1)n + (a^2-1)^{1/2}]. \tag{2.31}$$

Returning to (2.22) and employing the mean value theorem yields

$$Q_{n+1}(x) = \sum_{k=0}^{n+1} (-1)^{n+k} \frac{\omega'(\epsilon_k(x))}{\omega'(x_k)},$$

where $\epsilon_k(x)$ is between x and x_k, $k = 0, 1,..., n + 1$. Therefore

$$\| Q_{n+1} \| \leqslant \| \omega' \| \left\{ \frac{1}{|\omega'(-1)|} + \frac{1}{|\omega'(1)|} + \sum_{k=1}^{n} \frac{1}{|\omega'(x_k)|} \right\}.$$

Utilizing (2.26), (2.29), (2.30), and (2.31) then results in

$$\| Q_{n+1} \| \leqslant \frac{2(a+1)\, n + (a^2-1)^{1/2}}{(a+1)\, n + (a^2-1)^{1/2}} + \frac{2(a+1)\, n + (a^2-1)^{1/2}}{(a-1)\, n + (a^2-1)^{1/2}}$$

$$+ 2 \sum_{k=1}^{n} \left[\frac{2(a+1)\, n + (a^2-1)^{1/2}}{(a-1)\, n + (a^2-1)^{1/2}} \right]. \tag{2.32}$$

Thus $\| Q_{n+1} \| = O(n)$ for $n \geqslant 1$.

LEMMA 3. *Let g be defined by* (2.11). *Then for $n \geqslant 1$*

$$\left| \frac{g(x)}{n((a^2-1)^{1/2} + a)(x - x_i)} \right| = O(n), \tag{2.33}$$

$x \in I$, $i = 0, 1,..., n + 1$.

Proof. The mean value theorem implies there exists an $\epsilon_i(x)$ between x and x_i such that $g'(\epsilon_i(x)) = g(x)/(x - x_i)$. The definition of g and equations (2.24) and (2.26) imply that

$$\| g' \| \leqslant 2n[2(a+1)n + (a^2-1)^{1/2}]. \tag{2.34}$$

Utilizing (2.34) in the left side of expression (2.33) yields the result.

LEMMA 4. *Let a_{n+1} be the leading coefficient of the polynomial Q_{n+1} defined in* (2.9). *Then for $n \geqslant 1$*

$$\frac{2^{n-1}(a + (a^2-1)^{1/2})}{(a+1)} \leqslant | a_{n+1} | \leqslant 2^{n-1}(a + (a^2-1)^{1/2}). \tag{2.35}$$

Proof. Denote by $\bar{a}_{n+1}$ the leading coefficient of the polynomial $\bar{Q}_{n+1}$ defined by (2.20). Then comparisons of Q_{n+1} and $\bar{Q}_{n+1}$ reveal that

$$a_{n+1} = \sum_{k=0}^{n+1} \frac{(-1)^{n+k}}{\omega'(x_k)}$$

and

$$\bar{a}_{n+1} = \sum_{k=0}^{n+1} \frac{(-1)^{n+k}(x_k - a)}{\omega'(x_k)}$$

where $\omega'(x_k)$, $k = 0, 1,..., n+1$, is given by (2.25).

On the other hand, sgn $\omega'(x_k) = (-1)^{n+1-k}$, see [11, p. 35]. Therefore

$$|a_{n+1}| = \sum_{k=0}^{n+1} \frac{1}{|\omega'(x_k)|} \tag{2.36}$$

and

$$|\bar{a}_{n+1}| = \sum_{k=0}^{n+1} \frac{a - x_k}{|\omega'(x_k)|}. \tag{2.37}$$

But $a \geqslant 2$; therefore (2.36) and (2.37) imply that

$$|a_{n+1}| \leqslant |\bar{a}_{n+1}| \leqslant (a+1)|a_{n+1}|. \tag{2.38}$$

But (2.20) implies that

$$|\bar{a}_{n+1}| = 2^{n+1}(a + (a^2 - 1)^{1/2}).$$

This equality and (2.38) imply (2.35).

Lemmas 1–4 now facilitate the proof of Theorem 3.

Proof of Theorem 3. According to earlier observations we need only verify inequality (2.8). But from Lemma 1

$$q_{in}(x) = Q_{n+1}(x) - \frac{a_{n+1}g(x)}{2^{n-1}n((a^2-1)^{1/2}+a)(x-x_i)}.$$

Therefore

$$\| q_{in} \| \leqslant \| Q_{n+1} \| + \frac{|a_{n+1}|}{2^{n-1}} \left[\frac{\| g' \|}{n((a^2-1)^{1/2}+a)}\right].$$

The conclusions of Lemmas 2, 3, and 4 now combine to imply for all $n \geqslant 1$ that

$$\| q_{in} \| \leqslant \beta n, \tag{2.39}$$

for some positive constant β, $i = 0, 1,..., n+1$. To conclude the proof of theorem 3 we must show that there exists a positive constant α such that for $n \geqslant 1$

$$\max_{0 \leqslant i \leqslant n+1} \| q_{in} \| \geqslant \alpha n.$$

If $i = n + 1$ and $x = 1$, then (2.10) and (2.11) imply that

$$q_{n+1,n}(1) = -1 - \frac{2a_{n+1}[n(a^2-1)^{1/2}\, C_n(1) + (a-1)\, C_n'(1)]}{2^{n-1}n((a^2-1)^{1/2}+a)}.$$

Therefore

$$|q_{n+1,n}(1)| \geqslant \frac{2\,|\,a_{n+1}\,|\,[n(a^2-1)^{1/2} + (a-1)\,n^2]}{2^{n-1}n((a^2-1)^{1/2}+a)} - 1.$$

Now (2.35) implies that

$$|q_{n+1,n}(1)| \geqslant \frac{2}{n(a+1)}\,[n(a^2-1)^{1/2} + (a-1)\,n^2] - 1.$$

Therefore for $n \geqslant 1$ there exists an $\bar{\alpha}$ such that

$$\|\, q_{n+1,n} \,\| \geqslant \bar{\alpha} n.$$

Combining this result with (2.39) establishes (2.8), concluding the proof of the theorem.

Remark. Although $f(x) = 1/(x-a)$, $a \geqslant 2$ is the first nonpolynomial function for which the precise order of $M_n(f)$ is known, the authors conjecture for any function g with $g^{(n+1)}$ nonvanishing on I for n sufficiently large, that $M_n(g)$ will be of precise order $O(n)$. The primary difficulty in proving this assertion by the above technique stems from the lack of information regarding the distribution of the points in the extremal set. The above techniques may be applicable to other rational functions, see [12].

3. Behavior of $M_n(f)$

In this section an explicit example satisfying (1.3) is constructed and analyzed. As already mentioned in Section 1, Schmidt [10] has constructed a $g \in C(I)$ for which (1.3) is valid. However, the analysis in [10] is somewhat technical and requires the use of a theorem due to Wolibner [13] on polynomial interpolation. Because of this, the various degrees of the polynomials utilized in [10] to construct the function g for which (1.3) is valid cannot be explicitly exhibited. Consequently Schmidt's construction is basically an existence construction.

To effect the construction of an explicit example for which (1.3) holds, define the sequence $\{n_k\}_{k=0}^{\infty}$ by $n_0 = 1$, $n_1 = 3$, and $n_{k+1} = n_k{}^3$. Thus

$$n_k = 3^{3^{k-1}}, \qquad k \geqslant 1. \tag{3.1}$$

Now let

$$f(x) = \sum_{k=0}^{\infty} \alpha_k C_{n_k}(x), \qquad -1 \leqslant x \leqslant 1, \tag{3.2}$$

where

$$\alpha_k = \frac{1}{2^k (n_k!)^2}. \tag{3.3}$$

By employing an argument similar to that given in [6, p. 91] it can be shown that the f defined in (3.2) is the restriction of an entire function to the segment $[-1, 1]$ of the complex plane.

Now let

$$p_{n_k}(x) = \sum_{j=0}^{k} \alpha_j C_{n_j}(x).$$

Then

$$f(x) - p_{n_k}(x) = \sum_{j=k+1}^{\infty} \alpha_j C_{n_j}(x). \tag{3.4}$$

If $x_i = \cos(i\pi/n_{k+1})$, $i = 0, 1,..., n_{k+1}$, then for $j \geqslant k + 1$,

$$C_{n_j}(x_i) = \cos \frac{n_j i\pi}{n_{k+1}} = (-1)^i,$$

$i = 0,..., n_{k+1}$. Thus if $e_{n_k}(f)(x) = f(x) - p_{n_k}(x)$, then (3.4) implies that

$$e_{n_k}(f)(x_i) = (-1)^i \sum_{j=k+1}^{\infty} \alpha_k = (-1)^i \| e_{n_k} \|,$$

$i = 0, 1,..., n_{k+1}$. Since $e_{n_k}(x)$ alternates $n_{k+1} + 1$ times,

$$B_m(f)(x) = p_{n_k}(x), \qquad m = n_k ,..., n_{k+1} - 1.$$

Appealing to (2.1) with $\mathscr{P}_{n+1} = \Pi_n$, we have that

$$M_{n_k}^{-1}(f) = \inf_{p \in S(\Pi_{n_k})} \{ \max_{x \in E_{n_k+1}(f)} \operatorname{sgn}[f(x) - B_{n_k}(f)(x)]\, p(x)\}. \tag{3.5}$$

Let p be any polynomial in $S(\Pi_{n_k})$, and suppose that $| p(x^*)| = 1$, $x^* \in [-1, 1]$. Then for some $i^* \in \{1,..., n_{k+1}\}$, $x^* \in [x_{i^*-1}, x_{i^*}]$, where $E_{n_k+1}(f) = \{x_0, x_1,..., x_{n_{k+1}}\}$. Without loss of generality assume that

$$\operatorname{sgn}[f(x_{i^*}) - B_{n_k}(f)(x_{i^*})]\, p(x^*) = 1$$

(otherwise replace x_{i^*} by x_{i^*-1}). Now

$$|x_{i^*} - x_{i^*-1}| \leqslant \frac{\pi}{n_{k+1}} = \frac{\pi}{3^{3^k}}.$$

Then

$$\begin{aligned}\max_{x \in E_{n_k+1}(f)} \operatorname{sgn}[f(x) - B_{n_k}(f)(x)]\, p(x) &\geqslant \operatorname{sgn}[f(x_{i^*}) - B_{n_k}(f)(x_{i^*})]\, p(x_{i^*}) \\ &= 1 - |p(x_{i^*}) - p(x^*)| \\ &= 1 - |p'(\epsilon)|\, |x_{i^*} - x^*| \\ &\geqslant 1 - \frac{n_k^2}{n_{k+1}}\pi,\end{aligned}$$

where ϵ is between x_{i^*} and x^*. Recalling the definition of n_k, this inequality implies that

$$M_{n_k}^{-1}(f) \geqslant 1 - \frac{\pi}{3^{3^{k-1}}}.$$

Since for every n, $M_n(f) \geqslant 1$, this inequality implies that

$$\liminf_n M_n(f) = 1.$$

On the other hand,

$$e_m(f) = f(x) - B_m(f)(x) = f(x) - B_{n_k}(f)(x),$$

$m = n_k, \ldots, n_{k+1} - 1$. Therefore if $m = n_{k+1} - 1$, then the alternating set consists of precisely $n_{k+1} + 1$ points, namely the $n_{k+1} + 1$ extreme points of $C_{n_{k+1}}$. An argument similar to that used to prove Theorem 2 can now be employed to establish that there exists positive constants α and β such that

$$\alpha(n_{k+1} - 1) \leqslant M_{n_{k+1}-1}(f) \leqslant \beta(n_{k+1} - 1)$$

for all k sufficiently large. Therefore

$$\limsup_n M_n(f) = +\infty.$$

In terms of inequality (1.4), the above analysis establishes the existence of a function c (as described above (1.4)) satisfying $c(n_k) = 1$ and $c(n_{k+1} - 1) = n_{k+1} - 1$, for k sufficiently large. It would be of interest to discover the values of c for all natural numbers $n \geqslant N$.

The analysis of the present section leaves unanswered the question as to whether or not the sequence (1.2) can be bounded for any non-polynomial function, but does, in combination with the work of Schmidt [10], suggest that the key to resolving the question rests with proving or disproving that there exists a function f such that $e_n(f)$ has more than $n + 2$ extremal points for every n.

4. Observations and Conclusions

In the preceding sections we analyze for certain functions $f \in C(I)$ the behavior of the strong unicity constant as a function of changing dimension. The problems discussed at the ends of Sections 2 and 3 certainly merit further investigations.

In addition, the possible relationship between the strong unicity constant and the classical Lebesque constant [11, p. 90] needs investigating. The analysis of Section 3 suggests to these authors that for functions f with nonvanishing derivatives $f^{(n)}$, $n \geqslant N$, that an identifiable relationship between these two constants may exist.

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